CHAPTER-5 EIGENVALUES AND EIGENVECTORS

EIGENVALUES OR CHARACTERISTIC ROOTS: The eigenvalues of an $n \times n$ matrix A are the solutions of Characteristic Equation

$$\det(\lambda I - A) = 0.$$

Example. Find the eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$.

Solution: The characteristic equation of the matrix A is given by

$$\det(\lambda I - A) = 0$$

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} \lambda - 5 & 2 \\ -3 & \lambda \end{vmatrix} = 0$$

$$i.e., \ \lambda(\lambda - 5) + 6 = 0$$

$$i.e., \ \lambda^2 - 5\lambda + 6 = 0 \quad \text{(Characteristic Equation)}$$

$$i.e., (\lambda - 2)(\lambda - 3) = 0$$

$$i.e., \lambda = 2, 3$$

Hence, the eigenvalues (or characteristic roots) of matrix A are 2 & 3.

NOTE (1) The characteristic equation of an $n \times n$ matrix is of degree 'n'' and so an $n \times n$ matrix has at most $'n'$ distinct eigenvalues.

NOTE (2) The sum of eigenvalues of a square matrix is equal to its Trace and the product of eigenvalues of a square matrix is equal to its Determinant i.e.,

If $\lambda_1, \lambda_2, \lambda_3, \dots \dots \dots, \lambda_n$ are the eigenvalues of a square matrix A, then

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots \dots \dots + \lambda_n = Trace(A)$$

And $\quad (\lambda_1)(\lambda_2)(\lambda_3) \dots \dots \dots (\lambda_n) = |A|$

THEOREM: If 'A' is $n \times n$ triangular matrix (upper triangular, lower triangular or diagonal), then the eigenvalues of matrix 'A' are just the entries on the main diagonal of 'A'.

EIGENVECTORS: The eigenvectors corresponding to an eigenvalue λ of a matrix A are the non-zero vectors that satisfy the equation $(\lambda I - A)X = 0$.

Example: Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$.

Solution: The characteristic equation of the matrix A is given by

$$\det(\lambda I - A) = 0$$

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} \lambda - 5 & 2 \\ -3 & \lambda \end{vmatrix} = 0$$

$$i.e., \lambda(\lambda - 5) + 6 = 0$$

$$i.e., \lambda^2 - 5\lambda + 6 = 0 \text{ (Characteristic Equation)}$$

$$i.e., (\lambda - 2)(\lambda - 3) = 0$$

$$i.e., \lambda = 2, 3$$

The eigenvalues (or characteristic roots) of matrix A are 2 & 3.

The eigenvector corresponding to eigenvalue $\lambda = 2$ is the non-trivial solution of the equ.

$$(\lambda I - A)X = \mathbf{0}$$

$$i.e., \left(2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e., \begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives $3x_1 - 2x_2 = 0$ ……………….(1)

Let $x_2 = t$, then from (1) $3x_1 - 2t = 0$

$$i.e., x_1 = \frac{2}{3}t$$

Thus, the eigenvector corresponding to $\lambda = 2$ is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Similarly, we can find the Eigenvector corresponding to the eigenvalue $\lambda = 3$.

THEOREM: If k is a positive number, λ is an eigenvalue of a matrix 'A' and X is corresponding eigenvector, then λ^k is an eigenvalue of A^k and X is a corresp. eigenvector.

THEOREM: A square matrix A is Invertible iff $\lambda = 0$ is not an eigenvalue of A.

SIMILAR MATRICES: If 'A' and 'B' are square matrices, then we say that 'B' is similar to 'A' if there is an invertible matrix P such that $B = P^{-1}AP$.

NOTE: The similar matrices 'A' and $P^{-1}AP$ have same determinant, same rank, same nullity, same trace, same characteristic polynomial and same eigenvalues.

DIAGONALIZABLE: A square matrix 'A' is said to be Diagonalizable if it similar to some diagonal matrix; that is, if there is an invertible matrix 'P' such that $P^{-1}AP$ is diagonal. In this case, the matrix 'P' is said to diagonalize matrix 'A'.

THEOREM: If 'A' is an $n \times n$ matrix, the following statements are equivalent-

(i) Matrix 'A' is diagonalizable.
(ii) Matrix 'A' has 'n' linearly independent eigenvectors.

THEOREM: If an $n \times n$ matrix 'A' has 'n' distinct eigenvalues (the eigenvectors of 'A' are linearly independent), then matrix 'A' is Diagonalizable.

NOTE (1) A triangular matrix with distinct entries on main diagonal is Diagonalizable.

REVIEW OF COMPLEX NUMBERS: If $z = a + ib$ is a complex number, then

(i) $Re(z) = a$ and $Im(z) = b$ are called Real Part & Imaginary Part of z respectively.
(ii) $|z| = \sqrt{a^2 + b^2}$ is called the modulus (or absolute value) of z.
(iii) $\bar{z} = a - ib$ is called the complex conjugate of z.
(iv) $z\bar{z} = (a + ib)(a - ib)$
$= a^2 + b^2 = |z|^2$

Where i , called 'iota' has the property $i^2 = -1$ or $i = \sqrt{-1}$.

NOTE (1) Every vector in C^n can be split into Real & Imaginary parts as-

$$v = (a_1 + ib_1, a_2 + ib_2, \dots \dots \dots, a_n + ib_n) = (a_1, a_2, \dots \dots a_n) + i(b_1, b_2, \dots \dots., b_n)$$

& $$\bar{v} = (a_1 - ib_1, a_2 - ib_2, \dots \dots \dots, a_n - ib_n) = (a_1, a_2, \dots \dots a_n) - i(b_1, b_2, \dots \dots., b_n)$$

NOTE (2) If 'A' is a complex matrix, then *Re(A)* and *Im(A)* are the matrices formed from the real and imaginary parts of the entries of 'A' and

'$\bar{A}$' is the matrix formed by taking complex conjugate of each entry in 'A'.

Example: For the matrix $A = \begin{bmatrix} 1+3i & 2 \\ 3+i & 4-i \end{bmatrix}$, find $\bar{A}, Re(A), Im(A), Tr(A)$ & $\det(A)$.

Solution: $\bar{A} = \begin{bmatrix} 1-3i & 2 \\ 3-i & 4+i \end{bmatrix}$

$Re(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$Im(A) = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$

$Tr(A) = 1 + 3i + 4 - i = 5 + 2i$

$\det(A) = (1+3i)(4-i) - 2(3+i)$

$= 4 - i + 12i - 3i^2 - 6 - 2i$

$= -2 + 9i - 3(-1) = 1 + 9i \qquad (\because\ i^2 = -1)$

THEOREM: If u & v are vectors in C^n and if k is a scalar, then

(i) $\overline{(\bar{u})} = u$
(ii) $\overline{(ku)} = \bar{k}\bar{u}$
(iii) $\overline{(u+v)} = \bar{u} + \bar{v}$

THEOREM: If 'A' is an $m \times k$ complex matrix and 'B' is a $k \times n$ complex matrix, then

(i) $\overline{(\bar{A})} = A$
(ii) $\overline{(A^T)} = (\bar{A})^T$
(iii) $\overline{(AB)} = \bar{B}\bar{A}$

COMPLEX EUCLIDEAN INNER PRODUCT: If $u = (u_1, u_2, \dots \dots .., u_n)$ and $v = (v_1, v_2, \dots \dots .., v_n)$ are vectors in C^n, then

$$u.v = u_1\overline{v_1} + u_2\overline{v_2} + \cdots \dots \dots + u_n\overline{v_n}$$

Euclidean Norm on C^n is defined as $\|u\| = \sqrt{u.u} = \sqrt{u_1\overline{v_1} + u_2\overline{v_2} + \cdots \dots \dots + u_n\overline{v_n}}$

A vector u is called a Unit Vector in C^n if $\|u\| = 1$.

And two vectors u & v are said to be Orthogonal if $u.v = 0$.

Example: Find $u.v, v.u, \|u\|$ and $\|v\|$ for vectors $u = (1+i, i, 3-i)$ & $v = (1-i, 2, 4i)$.

Solution: We have

$$u.v = (1+i, i, 3-i).(1-i, 2, 4i)$$
$$= (1+i)\,\overline{(1-\imath)} + i\,\overline{(2)} + (3-i)\,\overline{(4\imath)}$$
$$= (1+i)(1+i) + 2i + (3-i)(-4i)$$
$$= 1 + i + i + i^2 + 2i - 12i + 4i^2$$
$$= 1 - 8i + 5i^2$$
$$= 1 - 8i + 5(-1) = -4 - 8i$$

Similarly, $v.u = (1-i, 2, 4i).(1+i, i, 3-i)$

$$= (1-i)\,\overline{(1+\imath)} + 2\,\overline{(\imath)} + (4i)\,\overline{(3-\imath)}$$
$$= (1-i)(1-i) - 2i + 4i(3+i) = -4 + 8i \qquad (\because\ i^2 = -1)$$

$$\|u\| = \sqrt{|1+i|^2 + |i|^2 + |3-i|^2}$$
$$= \sqrt{(1^2+1^2) + (0^2+1^2) + \{(3)^2 + (-1)^2\}} \qquad (\because\ |a+ib|^2 = a^2 + b^2)$$
$$= \sqrt{1+1+1+9+1} = \sqrt{13}$$

Similarly, $\|v\| = \sqrt{|1-i|^2 + |2|^2 + |4i|^2} = \sqrt{22}$

THEOREM: If A is a 2×2 matrix with real entries, then the Characteristic Equation of A is

$$\lambda^2 - [trace(A)]\lambda + \det(A) = 0.$$

THEOREM: If 'A' is a real symmetric matrix, then 'A' has real eigenvalues.

TRUE AND FALSE QUESTIONS

State whether the following statements are True or False-

1. Matrix C is diagonalizable if it is similar to a diagonal matrix B; that is, there exists an invertible matrix P and $B = PCP^{-1}$. **(T)**
2. If an $n \times n$ matrix A has n distinct eigenvalues, then A is Diagonalizable. **(T)**
3. If A is real symmetric matrix, then A has complex eigenvalues. **(F)**
4. A square matrix 'A' is Invertible iff $\lambda = 0$ is an eigenvalue of 'A'. **(F)**
5. The characteristic polynomial of 2×2 matrix A is of degree 3. **(F)**
6. The product of eigenvalues of a square matrix is same as its determinant. **(T)**
7. Given that the characteristic polynomial of a matrix A is $p(\lambda) = (\lambda-1)(\lambda+2)(\lambda-3)^2$, then $\det(A^{-1}) = -\frac{1}{18}$. **(T)**
8. The sum of eigenvalues of a square matrix is equal to its Trace. **(T)**
9. If 0, 1, and 2 are the eigen values of a matrix A, then $|A| = 0$. **(T)**
10. $(2, 1, 3)$ is the imaginary part of the complex vector $(2i+2, i+1, 3i-3)$. **(T)**

OBJECTIVE QUESTIONS

1. If '0' is an eigenvalue of a square matrix 'A' then 'A' is-
   a) Invertible
   b) Not Invertible
   c) An Identity matrix
   d) None of above

2. If {1,2,3} are the eigenvalues of a matrix, then its Trace and Determinant are-
   a) 3,3
   b) 4,4
   c) 5,5
   d) 6,6

3. The characteristic equation of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ is
   a) $\lambda^2 - 4\lambda + 1 = 0$
   b) $\boldsymbol{\lambda^2 - 4\lambda - 1 = 0}$
   c) $\lambda^2 + 4\lambda + 1 = 0$
   d) $\lambda^2 + 4\lambda - 1 = 0$

4. The eigenvalues of the matrix A^4 are, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 4 & 3 \end{bmatrix}$
   a) $\{1, -2, 3\}$
   b) $\{1, -16, 81\}$
   c) $\{\mathbf{1}, \mathbf{16}, \mathbf{81}\}$
   d) $\{1, -16, -81\}$

5. The characteristic polynomial of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$ is
   a) $\lambda^2 - 9 = 0$
   b) $\lambda^2 + 9 = 0$
   c) $\boldsymbol{\lambda^2 - 9}$
   d) $\lambda^2 + 9$

6. The eigenvalues of a matrix, $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 3 & -1 \end{bmatrix}$ are-
   a) {1,2,3}
   b) {1,2,2}
   c) {1,2,0}
   d) $\{\mathbf{1}, \mathbf{-1}, \mathbf{2}\}$

7. The sum of eigenvalues of matrix $A = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 1 & 0 \\ 1 & 3 & -3 \end{bmatrix}$ is
   a) 6
   b) 1
   c) 0
   d) 5

CHAPTER-6 INNER PRODUCT SPACE

INNER PRODUCT: An inner product on a real vector space V is a function that associates a real no. $< u, v >$ with each pair of vectors in V in such a way that following axioms are satisfied

(i) $< u, v > = < v, u >$
(ii) $< u + v, w > = < u, w > + < v, w >$
(iii) $< ku, v > = k < u, v >$
(iv) $< v, v > \geq 0$ and $< v, v > = 0$ iff $v = 0$.

EUCLIDEAN INNER PRODUCT OR STANDARD INNER PRODUCT ON R^n: If $u = (u_1, u_2, \dots \dots .., u_n)$ and $v = (v_1, v_2, \dots \dots .., v_n)$ are vectors in R^n, then

$$< u, v >= u_1v_1 + u_2v_2 + \cdots \dots \dots + u_nv_n$$

If V is a real inner product space, then the norm (or length) of a vector u in V is defined as

$$\|u\| = \sqrt{< u, u >}$$

And the distance between two vectors u & v is defined as

$$d(u, v) = \|u - v\| = \sqrt{< u - v, u - v >}$$

Example: If $u = (1, 0)$ and $v = (0, 1)$ are two vectors in R^2 with Euclidean inner product $< u, v >= u_1v_1 + u_2v_2$. Find $\|u\|$ and distance between u & v.

Solution: We have

$$\|u\| = \sqrt{< u, u >} = \sqrt{(1)(1) + (0)(0)} = 1$$

$$d(u, v) = \|u - v\| = \|(1, -1)\|$$

$$= \sqrt{< (1, -1), (1, -1) >}$$

$$= \sqrt{(1)(1) + (-1)(-1)} = \sqrt{2}$$

INNER PRODUCT ON M_{nn}: If U and V are $n \times n$ matrices , then $< U, V >= trace\ (U^TV)$.

If $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$, then $< U, V > = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.

Example: Find the value of k, for which the matrices $U = \begin{bmatrix} 9 & 4 \\ 2 & 6 \end{bmatrix}$ $and\ V = \begin{bmatrix} 2 & -2 \\ k & 4 \end{bmatrix}$ are orthogonal in the vector space $M_{2\times2}$ with usual inner product on $M_{2\times2}$.

Solution: If matrices U and V are orthogonal, then

$$< U, V >= 0$$

$$< \begin{bmatrix} 9 & 4 \\ 2 & -6 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ k & 4 \end{bmatrix} >= 0$$

$$i.e., (9)(2) + (4)(-2) + (2)(k) + (-6)(4) = 0$$

$$2k = 14\ \ i.e., k = 7$$

STANDARD INNER PRODUCT ON P_n

Let $p = a_0 + a_1x + \cdots \dots \dots \dots .. + a_nx^n$ and $q = b_0 + b_1x + \cdots \dots \dots \dots .. + b_nx^n$ are polynomials in P_n, then the standard inner product on P_n is defined as-

$$< p, q > = a_0b_0 + a_1b_1 + \cdots \dots \dots \dots . + a_nb_n$$

Norm of a polynomial p relative to this inner product is

$$\|p\| = \sqrt{< p, p >} = \sqrt{a_0{}^2 + a_1{}^2 + \cdots \dots .. + a_n{}^2}$$

ANGLE BETWEEN VECTORS: Let θ be the angle between u & v in a real inner product space, then

$$cos\theta = \frac{< u, v >}{\|u\|\|v\|}, \qquad 0 \leq \theta \leq \pi.$$

ORTHOGONALITY:

Two vectors u & v in inner product space are called Orthogonal if $< u, v > = 0$.

Example: Show that the vectors $u = (0, 2, 0)$, $v = (3, 0, 3)$, $w = (-4, 0, 4)$ form an orthogonal basis for R^3 with Euclidean Inner Product.

Solution: Here $u = (0, 2, 0)$, $v = (3, 0, 3)$, $w = (-4, 0, 4)$

Now $< u, v > = < (0, 2, 0), (3, 0, 3) > = (0)(3) + (2)(0) + (0)(3) = 0$

$< u, w > = < (0, 2, 0), (-4, 0, 4) > = (0)(-4) + (2)(0) + (0)(4) = 0$

$< v, w > = < (3, 0, 3), (-4, 0, 4) > = (3)(-4) + (0)(0) + (3)(4) = 0$

So the vectors u, v & w form an orthogonal set and hence these vectors are linearly independent and hence form a basis for R^3.

GENERALIZED THEOREM OF PYTHAGORAS:

If u & v are orthogonal vectors in an inner product space , then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

LEAST SQUARES SOLUTION OF LINEAR SYSTEMS:

For every linear system $AX = B$, the associated normal system $A^T AX = A^T B$ is consistent and all solutions of this system are least squares solutions of the system $AX = B$.

Example: Find the least squares solution of the system of linear equation $AX = B$,

$$\text{where } A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

OR

Find the least squares solution of the linear system

$$x - 2y = -3$$
$$2x + y = 2$$
$$2x = 1$$

Solution: The given system of equations in matrix form is $AX = B$,

$$\text{Where } A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \text{ \& } X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{We have } A^T A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\text{And} \quad A^T B = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

The associated normal system is given by $A^T AX = A^T B$

$$i.e., \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$i.e., \begin{bmatrix} 9x \\ 5y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Which gives $9x = 3$ and $5y = 8$

Hence, $x = \frac{1}{3}$, $y = \frac{8}{5}$ are the required least squares solution.

TRUE AND FALSE QUESTIONS

State whether the following statements are True or False-

1. If $u = (3, 4)$ is a vector in R^2, then the length of 'u' is 7. **(F)**
2. The norm of the vector $u = (3, 0, 4, 1)$ in R^4 is $\sqrt[3]{26}$. **(F)**
3. If u & v are unit orthogonal vectors in an inner product space, then $\|u + v\| \neq 0$. **(T)**
4. The inner product of a vector with itself (i.e.,$< u, u >$) can be negative real no. **(F)**
5. The inner product of a nonzero vector with itself is always a positive real no. **(T)**
6. If $u = (0, 1), v = (1, 0)$ & $k = 2$, then $< ku, v \geq 0$. **(T)**
7. If $u = (2, 1, 0, -3), v = (3, 0, 2, 2)$, then $< u, v >= 0$. **(T)**

OBJECTIVE QUESTIONS

1. Which of the following sets of vectors are orthogonal with respect to the inner product defined by $< u, v >= 3u_1v_1 + 2u_2v_2$ on R^2:
   a) (1, -1), (2, 2)
   b) (-1, 1), (2, -3)
   c) (-1, 1), (2, 3)
   d) (-1, 1), (-2, 3)

2. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on R^2:
   a) $(1, 2, -5)$ & $(-3, -1, 1)$
   b) $(\mathbf{1}, \mathbf{2}, -\mathbf{5})$ & $(-\mathbf{3}, -\mathbf{1}, -\mathbf{1})$
   c) $(1, 2, 0)$ & $(0, 1, 1)$
   d) $(0, 0, 1)$ & $(1, 0, 1)$

3. If angle between vectors u and v is zero such that $\|u\| = 2$, $\|v\| = 3$, then $< u, v >=$
   a) 5
   b) $\sqrt{5}$
   c) 6
   d) $\sqrt{6}$
4. If cosine value of angle between vectors u and v is $\frac{1}{2}$ and $\|u\| = 2$, $\|v\| = 3$, then $< u, v >=$
   a) 6
   b) 3
   c) 2
   d) 1
5. If $u = (-2, 1, 5)$, $v = (-1, -2, 2)$ and $k = 1$, then the value of $< ku, v >$ is
   a) 5
   b) -5
   c) $\mathbf{10}$
   d) -10

6. If $u = (-2, 1, 5)$, $v = (-1, -2, 2)$, then the value of $< 3u, 5v >$ is
   a) 10
   b) 15
   c) 150
   d) 80

7. If $p = 3x + 4x^2$ is a vector in the vector space P_2, then $||P|| =$
   a) 7
   b) 25
   c) $\sqrt{12}$
   d) 5

8. The values of k for which $u = (k, -4, 8)$ and $v = (k, k, -4)$ are orthogonal in Euclidean Inner Product Space R^3 are-
   a) $4, -8$
   b) $-4, -8$
   c) $\mathbf{8}, \mathbf{-4}$
   d) 4,8

9. Which of the following vectors in R^3 are orthogonal with respect to the Euclidean inner product?
   a) $(2, -3, -12)$ & $(3, -2, 0)$
   b) $(2, -3, -12)$ & $(3, -2, -1)$
   c) $(2, 3,\ 12)$ & $(3, -2, 1)$
   d) $(\mathbf{2}, \mathbf{-3}, \mathbf{-12})$ & $(\mathbf{3}, \mathbf{-2}, \mathbf{1})$

10. If $U = \begin{bmatrix} -2 & -1 \\ 4 & -5 \end{bmatrix}, V = \begin{bmatrix} 3 & 5 \\ -6 & -4 \end{bmatrix}$, then $< U, V >$ is equal to
    a) -15
    b) 7
    c) 5
    d) 1

CHAPTER-7 DIAGONALIZATION & QUADRATIC FORMS

ORHTOGONAL MATRICES: A square matrix A is said to be Orthogonal if its transpose is the same as its inverse i.e., if $A^T = A^{-1}$

OR equivalently if $AA^T = A^T A = I$

Example: Show that the matrix $A = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$ is orthogonal and hence find A^{-1}.

Solution: We have $A^T = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$

Now $AA^T = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

Hence, matrix A is orthogonal.

And $A^{-1} = A^T = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$.

THEOREM: The following statements are equivalent for an $n \times n$ matrix 'A'-

(i) 'A' is orthogonal.

(ii) The row vectors of 'A' form an orthonormal set in R^n with respect to Euclidean inner product.

(iii) The column vectors of 'A' form an orthonormal set in R^n with respect to Euclidean inner product.

PROPERTIES OF ORTHOGONAL MATRICES:

(1) If matrix A is Orthogonal, then $\det(A) = 1 \text{ } or -1$ but converse is not necessarily true.

(2) The inverse of an orthogonal matrix is Orthogonal.

(3) The product of orthogonal matrices is Orthogonal.

ORTHOGONAL DIAGONALIZATION:

If 'A' & 'B' are square matrices, then we say that 'A' & 'B' are orthogonally similar if there is an orthogonal matrix 'P' such that $P^T AP = B$.

If the matrix 'A' is orthogonally similar to some diagonal matrix, say $P^T AP = D$, then we say that 'A' is orthogonally diagonalizable and that 'P' orthogonally diagonalizes 'A'.

CONDITIONS FOR ORTHOGONAL DIAGONALIZABILITY

THEOREM: If ‘A’ is an $n \times n$ matrix, then the following statements are equivalent –

(i) ‘A’ is orthogonally diagonalizable.
(ii) ‘A’ has an orthonormal set of n eigenvectors.
(iii) ‘A’ is Symmetric.

SYMMETRIC MATRIX: A real square matrix A is said to be Symmetric if $A^T = A$.

SKEW-SYMMETRIC MATRIX: A real square matrix A is said to be Skew-Symmetric if $A^T = -A$.

PROPERTIES OF SYMMETRIC MATRICES

THEOREM: If ‘A’ is symmetric matrix, then

(i) The eigenvalues of ‘A’ are all real numbers.
(ii) Eigenvectors from different eigenspaces are orthogonal.

QUADRATIC FORMS:

If ‘A’ is symmetric $n \times n$ matrix and X is an $n \times 1$ column vector of variables, then we call the function $Q_A(X) = X^T AX$, the Quadratic form associated with ‘A’.

Matrix corresponding to quadratic form on R^2 i.e., $ax^2 + by^2 + 2cxy$ is given as $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$.

The matrix corresponding to quadratic form on R^3 i.e.,

$a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3$ is given as $\begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}$.

CONJUGATE TRANSPOSE OF A MATRIX: If A is a complex matrix, then the conjugate transpose of A denoted by $A^*$is defined as $A^* = (\bar{A})^T = \overline{(A^T)}$.

PROPERTIES OF CONJUGATE TRANSPOSE:

(i) $(A^*)^* = A$
(ii) $(A + B)^* = A^* + B^*$
(iii) $(kA)^* = \bar{k}A^*$
(iv) $(AB)^* = B^*A^*$

HERMITIAN MATRIX: A square complex matrix A is said to be Hermitian if $A^* = A$.

PROPERTIES:

(1) The eigenvalues of a Hermitian matrix are real numbers.
(2) If ‘A’ is a Hermitian matrix, then eigenvectors from different eigenspaces are orthogonal.

SKEW-HERMITIAN MATRIX: A square complex matrix A is said to be Skew Hermitian if $A^* = -A$.

UNITARY MATRIX: A square complex matrix A is said to be Unitary if its conjugate transpose is the same as its inverse i.e., if $A^* = A^{-1}$

or equivalently if $AA^* = A^*A = I$.

Example: Show that the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is Unitary and hence find A^{-1}.

Solution: We have $A^* = (\bar{A})^T$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now $AA^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Hence A is a Unitary matrix and $A^{-1} = A^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

THEOREM: If ‘A’ is an $n \times n$ matrix with complex entries, then following are equivalent-

(i) ‘A’ is Unitary.

(ii) $\|AX\| = \|X\|$, $\forall X \in C^n$

(iii) $AX.AY = X.Y$, $\forall X, Y \in C^n$

(iv) The column vectors of ‘A’ form orthonormal set in C^n w.r.t. Complex Euclidean inner product.

(v) The row vectors of ‘A’ form orthonormal set in C^n w.r.t. Complex Euclidean inner product.

SKEW- SYMMETRIC AND SKEW-HERMITIAN MATRIX

A square matrix ‘A’ with real entries is defined to be Skew-symmetric if $A^T = -A$.

A skew- symmetric matrix must have zeros on main diagonal and each entry off the main diagonal must be the negative of its mirror image about main diagonal.

The example of skew symmetric matrix is $= \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$.

A square complex matrix ‘A’ is said to be Skew-Hermitian if $A^* = -A$.

A skew-hermitian matrix must have zeros or pure imaginary numbers on main diagonal and each entry off the main diagonal must be the negative of complex conjugate of its mirror image about the main diagonal.

The example of skew-hermitian matrix is $A = \begin{bmatrix} 2i & -1-i & 2 \\ 1-i & i & i \\ -2 & i & 0 \end{bmatrix}$.

TRUE AND FALSE QUESTIONS

State whether the following statements are True or False-

1. If a matrix ‘A’ is orthogonal, then $det(A) = 1 \ or \ -1$. **(T)**
2. If a matrix ‘A’ is orthogonal, then $(\det A)^2 = 1$. **(T)**
3. If determinant of a matrix is $1 \ or \ -1$, then the matrix is orthogonal. **(F)**
4. The rows and columns of an orthogonal matrix are orthogonal. **(T)**
5. The inverse of an orthogonal matrix is not necessarily orthogonal. **(F)**
6. If A is Orthogonal, then kA is also orthogonal for any scalar k. **(F)**
7. A square matrix ‘A’ is Orthogonal, if $A^* = A$. **(F)**
8. A square matrix ‘A’ is Unitary, if $A^* = A$. **(F)**
9. A square complex matrix A is Unitary if its conjugate transpose equals its inverse. **(T)**
10. In case of real matrices, Hermitian and Symmetric matrices are same. **(T)**
11. The matrix A= $\begin{bmatrix} 1 & 2-i & 1+i \\ 2+i & 2 & 1+2i \\ 1-i & 1+2i & 3 \end{bmatrix}$ is Hermitian. **(F)**

OBJECTIVE QUESTIONS

1. If $3x^2 - 4y^2 - 4xy$ be the quadratic form, then associated symmetric matrix will be
   a) $\begin{bmatrix} 3 & -4 \\ -4 & -4 \end{bmatrix}$
   b) $\begin{bmatrix} -4 & -4 \\ -4 & 3 \end{bmatrix}$
   c) $\begin{bmatrix} 3 & 2 \\ 2 & -4 \end{bmatrix}$
   d) $\begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{-4} \end{bmatrix}$

2. If $x^2 - 4y^2 + 3z^2 + 2xy + 4yz - 6zx$ be the quadratic form, then the associated symmetric matrix will be
   a) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & -4 & 2 \\ 3 & 2 & 3 \end{bmatrix}$
   b) $\begin{bmatrix} 1 & 1 & -3 \\ 1 & -4 & -2 \\ -3 & -2 & 3 \end{bmatrix}$
   c) $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{1} & \mathbf{-4} & \mathbf{2} \\ \mathbf{-3} & \mathbf{2} & \mathbf{3} \end{bmatrix}$
   d) $\begin{bmatrix} 1 & -1 & 3 \\ -1 & -4 & 2 \\ 3 & 2 & 3 \end{bmatrix}$

3. For which value of a & b, the matrix $\begin{bmatrix} 1 & 2-i & 1+i \\ a & 2 & 1+2i \\ 1-i & b & 3 \end{bmatrix}$ is Hermitian?
   a) $a = 2 + i, b = 1 + 2i$
   b) $a = 2 - i, b = 1 - 2i$
   c) $\boldsymbol{a = 2 + i, b = 1 - 2i}$
   d) $a = 2 - i, b = 1 + 2i$

4. If a square matrix A is such that $A^{-1} = A^*$, then matrix A is-
   a) Hermitian
   b) Skew Hermitian
   c) Unitary
   d) None
5. A complex square matrix A is said to be Hermitian matrix, if
   a) $AA^T = I$
   b) $A^T = A$
   c) $\boldsymbol{(\bar{A})^T = A}$
   d) $A^{-1} = A$

6. The eigenvalues of Hermitian matrix are-
   a) Complex only
   b) Complex & Real both
   c) Always Real
   d) Always Zero

CHAPTER-8 LINEAR TRANSFORMATION

GENERAL LINEAR TRANSFORMATION: If $T: V \to W$ is a function from a vector space V to a vector space W, then T is called a Linear Transformation from V to W if the following two properties hold for all vectors u & v in V and for all scalars k –

(i) $T(ku) = kT(u)$ [Homogeneity Property]
(ii) $T(u+v) = T(u) + T(v)$ [Additive Property]

OR

In combination of (i) & (ii), $T(k_1v_1 + k_2v_2) = k_1T(v_1) + k_2T(v_2)$, where $v_1, v_2 \in V$ and k_1, k_2 are scalars.

THEOREM: If $T: V \to W$ is a Linear Transformation, then

(i) $T(0) = 0$
(ii) $T(u - v) = T(u) - T(v)$, for all u & v in V

Example: Check whether the map $T: R^2 \to R^2$ given by $T(x,y) = (xy, x)$ is linear or not.

Solution: Let $u = (x_1, y_1)$, $v = (x_2, y_2) \in R^2$, so $u + v = (x_1 + x_2, y_1 + y_2)$

$$\begin{aligned} T(u+v) &= T(x_1+x_2, y_1+y_2) \\ &= \big((x_1+x_2)(y_1+y_2), x_1+x_2\big) \\ &= (x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1, x_1+x_2\ ) \qquad \text{……….(1)} \end{aligned}$$

$$\begin{aligned} T(u) + T(v) &= T(x_1, y_1) + T(x_2, y_2) \\ &= (x_1y_1, x_1) + (x_2y_2, x_2) \\ &= (x_1y_1 + x_2y_2, x_1 + x_2) \qquad \text{……….(2)} \end{aligned}$$

From (1) & (2), it is clear that $T(u+v) \neq T(u) + T(v)$

Therefore, T is not Linear.

Ex: Show that the function $T: R^2 \to R^2$ given by $T(x,y) = (y, x)$ is a linear transformation.

Solution: First we will show that $T(u+v) = T(u) + T(v)$

Let $u = (x_1, y_1)$, $v = (x_2, y_2) \in R^2$, so $u + v = (x_1 + x_2, y_1 + y_2)$

$$\begin{aligned} T(u+v) &= T(x_1+x_2, y_1+y_2) \\ &= \big((y_1+y_2), (x_1+x_2)\big) \qquad \text{…….(1)} \end{aligned}$$

And
$$\begin{aligned} T(u) + T(v) &= T(x_1, y_1) + T(x_2, y_2) \\ &= (y_1, x_1) + (y_2, x_2) \\ &= \big((y_1+y_2), (x_1+x_2)\big) \qquad \text{……..(2)} \end{aligned}$$

From (1) & (2), it is clear that $T(u+v) = T(u) + T(v)$

Now
$$\begin{aligned} T(ku) &= T\big(k(x_1, y_1)\big) \\ &= T(kx_1, ky_1) \\ &= (ky_1, kx_1) \\ &= k(y_1, x_1) \\ &= kT(u) \end{aligned}$$

Therefore, T is a Linear transformation.

Some Other Examples:

Example (1) Matrix Transformations: $T_A: R^n \to R^m$ is a linear transformation.

Example (2) Zero Transformation: The mapping $T: V \to W$ such that $T(v) = 0, \forall\, v \in V$ is a linear transformation.

Example (3) Identity Operator: The mapping $I: V \to V$ defined by $I(v) = v, \forall\, v \in V$ is a linear transformation.

Example (4) Dilation and Contraction Operators: If V is a vector space and k is a scalar, then $T: V \to V$ given by $T(x) = kx$ is a linear operator on V.

Example (5) Transformations on Matrix Spaces: Let M_{nn} be the vector space of all $n \times n$ matrices, then-

(i) $T_1(A) = A^T$ is a linear transformation.
(ii) $T_2(A) = \det(A)$ is not linear.

Example (6) Translation is not Linear: If x_0 is a fixed non-zero vector in R^2, then the transformation $T(x) = x + x_0$ is not linear.

Finding Linear Transformations from Images of Basis Vectors:

THEOREM: Let $T: V \to W$ is a linear transformation, where V is finite dimensional. If $S = \{v_1, v_2, \dots \dots \dots, v_n\}$ is a basis for V, then the image of any vector v in V can be expressed as $T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots \dots \dots \dots \dots \dots \dots + c_n T(v_n)$

where $c_1, c_2, \dots \dots c_n$ are coeff. required to express v as a linear combination of vectors in S.

Example: Let $S = \{v_1, v_2\}$ be the basis for R^2 where $v_1 = (1,1), v_2 = (1,0)$ and $T: R^2 \to R^2$ be the linear transformation such that $(v_1) = (1,2),\ T(v_2) = (3,0)$, then find $T(x,y)$.

Solution: First we need to express (x,y) as a linear combination of v_1 & v_2.

Let $(x,y) = c_1 v_1 + c_2 v_2$ ……….(1)
$i.e.,$ $(x,y) = c_1(1,1) + c_2(1,0)$
$i.e.,$ $(x,y) = (c_1 + c_2, c_1)$

Equating both sides,
$c_1 + c_2 = x$ ……….(2)
$c_1 = y$ ……….(3)
Solving (2) & (3), we get $c_1 = y$ & $c_2 = x - y$
Since T is linear,

$$\begin{aligned} T(x,y) &= c_1 T(v_1) + c_2 T(v_2), \quad \text{using equation (1)} \\ &= c_1(1,2) + c_2(3,0) \\ &= (c_1, 2c_1) + (3c_2, 0) \\ &= (c_1 + 3c_2, 2c_1) \\ &= [y + 3(x-y), 2y] \\ &= (3x - 2y, 2y) \end{aligned}$$

, which is the required formula for $T(x,y)$.

KERNEL AND RANGE OF LINEAR TRANSFORMATIONS: If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into '0' is called the Kernel of T and is denoted by $\ker(t)$. The set of all vectors in W that are images under T of at least one vector in V is called Range of T and is denoted by $R(t)$.

Example (1) Kernel and Range of Zero Transformation: $\ker(t) = V, \;\; R(t) = \{0\}$.

Example (2) Kernel and Range of Identity Operator: $\ker(I) = \{0\}, \;\; R(I) = V$.

PROPERTIES OF KERNEL AND RANGE: If $T: V \to W$ is a linear transformation, then

(i) The kernel of T is a subspace of V.
(ii) The range of T is a subspace of W.

RANK AND NULLITY OF LINEAR TRANSFORMATIONS: Let $T: V \to W$ is a linear transformation. If the range of T is finite dimensional, then its dimension is called the Rank of T; and if the kernel of T is finite dimensional, then its dimension is called the Nullity of T.

The Rank of T is denoted by rank(t) and Nullity of T is denoted by nullity(t).

THEOREM: If $T: V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then $rank(t) + nullity(t) = n$.

NOTE: If $T_A: R^n \to R^m$, then $rank(T_A) + nullity(T_A) = n$

ONE-TO-ONE LINEAR TRANSFORMATION: If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be One-to-One if T maps distinct vectors in V into distinct vectors in W.

ONTO LINEAR TRANSFORMATION: If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be Onto if every vector in W is the image of at least one vector in V.

THEOREM: If V is a finite-dimensional vector space and if $T: V \to V$ is a linear operator, then the following statements are equivalent-

(i) T is One-to-One.
(ii) $\ker(t) = \{0\}$.
(iii) T is Onto [i.e., $(t) = V$].

DIMENSION AND LINEAR TRANSFORMATIONS: There are two important facts about a linear transformation $T: V \to W$ in the case where V & W are finite-dimensional-

(i) If $\dim(W) < \dim(V)$, then T cannot be One-to-One.
(ii) If $\dim(V) < \dim(W)$, then T cannot be Onto.

ISOMORPHISM: If a linear transformation $T: V \to W$ is both One-to-One & Onto, then T is said to be an ISOMORPHISM and vector spaces V & W are said to be Isomorphic.

THEOREM: Every real n-dimensional vector space is Isomorphic to R^n.

Example (1) The linear transformation $T: P_{n-1} \to R^n$ defined by

$a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \to (a_0, a_1, \dots \dots, a_n)$ is a Natural Isomorphism from P_{n-1} to R^n.

Example (2) The transformation $T: M_{22} \to R^4$ defined by

$$T\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (a_1, a_2, a_3, a_4) \text{ is Natural Transformation from } M_{22} \to R^4.$$

NOTE: The vector space M_{mn} of $m \times n$ matrices with real entries is Isomorphic to R^{mn}.

COMPOSITIONS OF LINEAR TRANSFORMATIONS:

If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, then the Composition of T_2 with T_1, defined by $T_2 o T_1$ is the function from U to W defined by

$$(T_2 o T_1)(u) = T_2(T_1(u)), \text{ where } u \text{ is a vector in } U.$$

INVERSE LINEAR TRANSFORMATIONS

If T is One-to-One, then each vector w in $R(t)$ is the image of a unique vector v in V. This uniqueness allows us to define a new function, called the Inverse of T and denoted by T^{-1}, that maps w back into v.

THEOREM: If $T_1: U \to V$ and $T_2: V \to W$ are one-to-one linear transformations, then

(i) $T_2 o T_1$ is one-to-one.

(ii) $(T_2 o T_1)^{-1} = T_1^{-1} o\, T_2^{-1}$

TRUE AND FALSE QUESTIONS

State whether the following statements are True or False-

1. If $T: V \to V$ is an operator such that $T(v) = 2, \forall\, v \in V$, then T is linear. **(F)**
2. The function $T: R^2 \to R^3$ given by $T(x, y) = (2x + 3y, 4y - x - 1, x)$ is linear. **(F)**
3. The function $T: R^2 \to R^2$ given by $T(x, y) = (2x + 3y, 4y - 1)$ is linear. **(F)**
4. If 'T' is translation operator, then it is linear. **(F)**
5. If $T: V \to W$ is an isomorphism, then $\ker(T) = \{0\}$. **(T)**
6. If $T: V \to W$ is a one to one linear transformation, then $\ker(T) = \{0\}$. **(T)**
7. If $T: V \to W$ is a linear transformation, then $\ker(T)$ is a subspace of W. **(F)**

OBJECTIVE QUESTIONS

1. Let $T: R^2 \to R^2$ be a linear operator given by $T(x, y) = (y - x,\ -2x + 2y)$. Which of the following vector is in $Ker\ T$?
   a) $(-1, 2)$
   b) $(-1, 1)$
   c) $\boldsymbol{(1, 1)}$
   d) $(1, -1)$

2. If $T: V \to W$ be a linear transformation, then $\ker(T)$ & $range\ (T)$ are subspaces of vector space(s)
   a) V
   b) W
   c) $\boldsymbol{V}$ and $\boldsymbol{W}$ respectively.
   d) W and V respectively.

3. Which of the following transformations $T: R^2 \to R$ is a linear transformation?
   a) $T(x, y) = 1$
   b) $\boldsymbol{T(x, y) = 2x + y}$
   c) $T(x, y) = x + 1$
   d) $T(x, y) = x^2$

4. Let $T: R^5 \to R^4$ is a linear transformation with rank 3, then no. of basis elements in the kernel of T is-
   a) 1
   b) 2
   c) 3
   d) 4

5. If $T: M_{33} \to R^8$ is a linear transformation with rank 3, then Nullity of T is-
   a) 2
   b) 3
   c) 4
   d) 6

6. Let $T_1(x, y) = (y, x)$ and $T_2(x, y) = (x + y, x - y)$, then $T_2 o T_1(x, y) =$
   a) $(x + y, x - y)$
   b) $(x - y, x + y)$
   c) $\boldsymbol{(x + y, y - x)}$
   d) $(x + y, -x - y)$

CHAPTER-9 NUMERICAL METHODS

LU-DECOMPOSITION: A factorization of a square matrix A as $A = LU$, where L is Lower triangular & U is Upper triangular matrix, is called an LU-Decomposition of A.

To construct an LU-Decomposition for 2×2 matrix , we can take

$$L = \begin{bmatrix} 1 & 0 \\ l_1 & 1 \end{bmatrix} \ \& \ U = \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}.$$

NOTE: **Not every square matrix has an LU-Decomposition**. However, we will see that if it is possible to reduce a square matrix A to row echelon form by Gaussian elimination without performing any row interchanges, then matrix A will have an LU-Decomposition, though **it may not be unique**.

Example: Find LU-decomposition of the matrix $A = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}$.

Solution: Let $LU = A$

$$i.e., \begin{bmatrix} 1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}$$

$$i.e., \begin{bmatrix} u_1 & u_2 \\ l_1 u_1 & l_1 u_2 + u_3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}$$

Equating both sides, we get

$u_1 = 5, u_2 = -1$

$l_1 u_1 = -1$

$i.e., l_1(5) = -1 \Rightarrow \ l_1 = -\frac{1}{5}$

and $l_1 u_2 + u_3 = -1$

$i.e., \left(-\frac{1}{5}\right)(-1) + u_3 = -1 \Rightarrow \ u_3 = -\frac{6}{5}$

Hence, $L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{5} & 1 \end{bmatrix}$ $and\ U = \begin{bmatrix} 5 & -1 \\ 0 & -\frac{6}{5} \end{bmatrix}$

Which is the required $LU -$ decomposition.

DOMINANT EIGENVALUES: If the distinct eigenvalues of a matrix A are $\lambda_1, \lambda_2, \dots \dots, \lambda_k$ and $|\lambda_1|$ is larger than $|\lambda_2|, |\lambda_3|, \dots \dots \dots, |\lambda_k|$; then λ_1 is called a Dominant Eigenvalue of A. Any eigenvector corresponding to a dominant eigenvalue is called dominant eigenvector of A

NOTE: Some matrices have dominant eigenvalues and some do not have.

For example, if distinct eigenvalues of a matrix are $\lambda_1 = -4,\ \lambda_2 = -2,\ \lambda_3 = 1,\ \lambda_4 = 3$ then $\lambda_1 = -4$ is dominant since $|\lambda_1| = 4$ is greater than absolute values of all other eigenvalues.

But if the distinct eigenvalues of a matrix are $\lambda_1 = 7,\ \lambda_2 = -7,\ \lambda_3 = -2,\ \lambda_4 = 5$ then $|\lambda_1| = |\lambda_2| = 7$so there is no eigenvalue whose absolute value is greater than the absolute value of all the other eigenvalues.

THEOREM 1 If A is an $m \times n$ matrix, then

(i) The matrices A and $A^T A$ have the same null space.
(ii) The matrices A and $A^T A$ have the same row space.
(iii) The matrices A and $A^T A$ have the same column space.
(iv) The matrices A and $A^T A$ have the same rank.

THEOREM 2 If A is an $m \times n$ matrix, then

(i) $A^T A$ is orthogonally diagonalizable.
(ii) The eigenvalues of $A^T A$ are non-negative.

SINGULAR VALUES: If A is an $m \times n$ matrix and if $\lambda_1, \lambda_2, \dots \dots, \lambda_n$ are eigenvalues of symmetric matrix $A^T A$, then the numbers $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots \dots \sqrt{\lambda_n}$ are called the Singular values of matrix A.

Example: Find the Singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: We have

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation of $A^T A$ is $\det(\lambda I - A^T A) = 0$

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix}\right) = 0$$

$$(\lambda - 2)(\lambda - 2) - (-1)(-1) = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda = 3, 1$$

So the eigenvalues of $A^T A$ are $\lambda_1 = 3$ & $\lambda_2 = 1$

and singular values of 'A' are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$;

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

TRUE AND FALSE QUESTIONS

State whether the following statements are True or False-

1. Every matrix has dominant eigen value. **(F)**
2. Every square matrix need not have LU-Decomposition. **(T)**
3. The LU $-$decomposition of a matrix is unique. **(F)**
4. If A is an $m \times n$ matrix, then $A^T A$ is an $m \times m$ matrix. **(F)**
5. If A is an $m \times n$ matrix, then the eigenvalues of $A^T A$ cannot be negative. **(T)**

OBJECTIVE QUESTIONS

1. Which of the following set of eigenvalues has a dominant eigenvalue-
   a) $\{-10,0,1,10\}$
   b) $\{5,-5,2,3\}$
   c) $\{\mathbf{-4}, \mathbf{-3}, \mathbf{0}, \mathbf{1}\}$
   d) None
2. Which of the following is dominant eigenvalue of the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 2 & 0 & 2 & -4 \end{bmatrix}$.
   a) 3
   b) $\mathbf{-4}$
   c) 5
   d) No dominant eigen value
3. The singular values of a 3×3 matrix A are 2, $\sqrt{5}$ and $2\sqrt{2}$. The corresponding eigenvalues of the matrix B, where $B = A^T A$ are
   a) 2, $\sqrt{5}$ and $2\sqrt{2}$
   b) 2, 5 and 8
   c) 4, 5 and 8
   d) None
4. If $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ be a matrix where $B = A^T A$, then singular values of A are-
   a) 1, 3
   b) 3, 2
   c) 1,$\sqrt{3}$
   d) 3, $\sqrt{3}$
5. What are the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.
   a) 1 & 3
   b) 0,1 & 3
   c) 1 & $\sqrt{3}$
   d) 0, 1 & $\sqrt{3}$
6. If $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ be a matrix where $B = A^T A$, then singular values of A are
   a) $\{1, 4, 9\}$
   b) $\{1, 4, 0\}$
   c) $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$
   d) $\{0, 4, 9\}$

CHAPTER-10 LINEAR PROGRAMMING PROBLEM(LPP)

GENERAL LINEAR PROGRAMMING PROBLEM IN TWO VARIABLES:

Find the values of x_1 & x_2 that optimize (either maximize or minimize)

$$z = c_1x_1 + c_2x_2 \qquad \text{[Linear Objective Function]}$$

Subject to Linear Constraints $a_{11}x_1 + a_{12}x_2 \ (\leq, \geq \ or\ =) b_1$

$$a_{21}x_1 + a_{22}x_2 \ (\leq, \geq \ or\ =) b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$a_{m1}x_1 + a_{m2}x_2 \ (\leq, \geq \ or\ =) b_m$$

And $x_1 \geq 0,\ x_2 \geq 0$ [Non-Negative Constraints]

NOTE (1) A pair of values (x_1, x_2) that satisfy all the constraints is called a **Feasible Solution**. The set of all feasible solutions determines a subset of x_1x_2-plane called the feasible region. A feasible solution that optimizes the objective function is called an **Optimal Solution**.

NOTE (2) The feasible region of an LPP has a boundary consisting of a finite number of straight line segments. If the feasible region can be enclosed in a sufficiently large circle, it is called **Bounded**; otherwise it is called Unbounded.

If the feasible region is empty (contains no points), then the constraints are **Inconsistent** and the LPP has no solution.

Those boundary points of a feasible region that are intersections of two of the straight line boundary segments are called **Extreme points (or Corner points)**.

THEOREM: If the feasible region of an LPP is non-empty and bounded, then the objective function attains both a maximum and a minimum value and these occur at extreme points of the feasible region. If the feasible region is Unbounded, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at an extreme point.

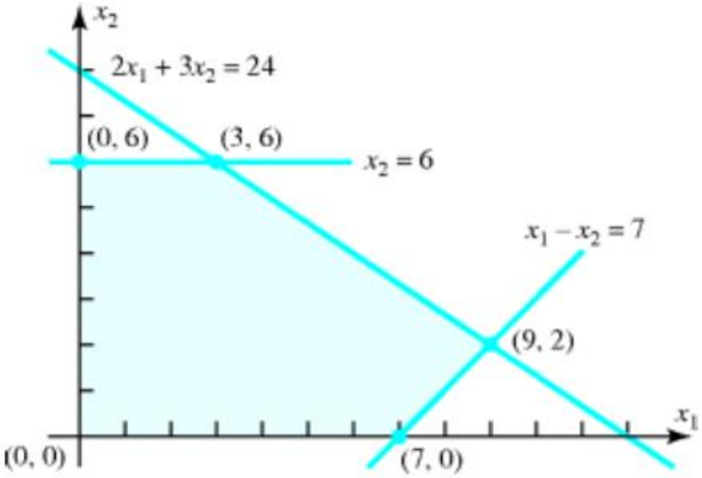
Example: Solve the following LPP by Graphical method-

$$\max z = x_1 + 3x_2$$

subject to:

$$2x_1 + 3x_2 \leq 24$$
$$x_1 - x_2 \leq 7$$
$$x_2 \leq 6$$
$$and\ x_1, x_2 \geq 0$$

Solution: In Fig, we have drawn the feasible region of this problem.



Since the feasible region is bounded, the maximum value of z is attained at one of the extreme points. The values of objective function at five extreme points are given in the following table:

Extreme Points (x_1, x_2)	O(0,0)	A(0,6)	B(3,6)	C(9,2)	D(7,0)
$z = x_1 + 3x_2$	0	18	21	15	7

From the Table, the maximum value of z is 21 which is attained at $x_1 = 3$ & $x_2 = 6$.

TRUE AND FALSE QUESTIONS

State whether the following statements are True or False-

1. In LPP, all variables are restricted to positive values only. **(F)**
2. In LPP, a non-linear objective function is to be optimized. **(F)**
3. The graphical method can be used to solve LPP with any No. of unknown variable. **(F)**
4. One of the quickest ways to plot a constraint is to find the two points where the constraint crosses the axes, and draw a straight line between these points. **(T)**
5. No LPP with an unbounded feasible region has a solution. **(F)**

OBJECTIVE QUESTIONS

1. The valid Objective Function for a LPP is-
   a) $\max(x, y)$
   b) $\min(x^2 + y^2)$
   c) $\boldsymbol{\min\left(x + y - \frac{1}{2}z\right)}$
   d) None

2. Which of the following constraints is not linear?
   a) $7x - 6y \leq 45$
   b) $x - y + z \geq 25$
   c) $\boldsymbol{xy - y = 5}$
   d) $x - \frac{1}{3}y = 5$
3. In maximization problem, optimal solution occurring at corner point yields the
   a) Mean values of z
   b) Lowest value of z
   c) Mid values of z
   d) Highest value of $\boldsymbol{z}$

4. In linear programming, objective function and constraints are
   a) Quadratic and linear respectively
   b) Linear and quadratic respectively
   c) Both are quadratic
   d) Both are linear

5. The feasible region
   a) Represents all values of each constraint
   b) May range over all positive or negative values of only one decision variable.
   c) Is an area bounded by the collective constraints and represents all permissible combinations of the decision variables
   d) Is defined by the objective function

SOME IMPORTANT FORMULAE FROM CHAPTER-1 TO CHAPTER-4

CHAPTER-2

FORMULA FOR 2 × 2 MATRIX:

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is Invertible iff $ad - bc \neq 0$ $(i.e., |A| \neq 0)$

And $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The quantity $ad - bc$ is called the Determinant of 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$i.e., \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

TRIANGULAR MATRICES: A matrix that is either Upper-triangular or Lower- triangular is called the Triangular Matrix.
NOTE(1) The diagonal matrices are both Upper triangular and Lower triangular.
NOTE(2) A triangular matrix is Invertible iff its diagonal entries are all non-zero.

PROPERTIES OF TRIANGULAR MATRICES:

(1) The transpose of a Lower triangular matrix is Upper triangular matrix and
The transpose of an Upper triangular matrix is Lower triangular matrix.

(2) The product of Lower triangular matrices is Lower triangular matrix and
The product of Upper triangular matrices is Upper triangular matrix.

(3) The Inverse of an invertible Lower triangular matrix is Lower triangular matrix and
The Inverse of an invertible Upper triangular matrix is Upper triangular matrix.

INVERTIBILTY OF SYMMETRIC MATRICES: In general, a Symmetric matrix need not be Invertible. For example, a diagonal matrix with a zero on main diagonal is Symmetric but not Invertible.

THEOREM: If A is an invertible symmetric matrix, then A^{-1} is symmetric.

NOTE: The products AA^T and A^TA are always symmetric.

DETERMINANT OF A TRIANGULAR MATRIX: If A is an $n \times n$ Triangular matrix (Upper triangular, lower triangular or diagonal), then $\det(A)$ is the product of entries on main diagonal of the matrix, that is,

$$\det(A) = a_{11}a_{22} \dots \dots \dots \dots \dots \dots a_{nn}$$

THEOREM: Let A be a square matrix. If A has a row of zeroes or a column of zeroes, then $\det(A) = 0$.

THEOREM: Let A be a square matrix, then $\det(A) = \det(A^T)$.

THEOREM: If A be a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

BASIC PROPERTIES OF DETERMINANTS:

(1) If A is an $n \times n$ matrix and k is any scalar, then $\det(kA) = k^n \det(A)$.
(2) If A and B are square matrices of same size, then $\det(AB) = \det(A) . \det(B)$.

DETERMINANT TEST FOR INVERTIBILTY (THEOREM) : A square matrix A is Invertible iff $\det(A) \neq 0$.

NOTE: If a matrix A is Invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

CHAPTER-3

Vectors Whose Initial Point is Not at the Origin

If $\overrightarrow{P_1P_2}$ denotes a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Similarly, the components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ are

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Operations of Vectors on R^n

If $\mathbf{v} = (v_1,\ v_2, \dots \dots . . , v_n)$ and $\mathbf{w} = (w_1,\ w_2, \dots \dots . . , w_n)$ are vectors in R^n, and if k is any scalar, then

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1,\ v_2 + w_2, \dots \dots . . , v_n + w_n)$
- $k\mathbf{v} = (kv_1,\ kv_2, \dots \dots . . , kv_n)$
- $-\mathbf{v} = (-v_1,\ -v_2, \dots \dots . . , -v_n)$
- $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1,\ w_2 - v_2, \dots \dots . . ,\ w_n - v_n)$

Norm of a Vector

If $\mathbf{v} = (v_1,\ v_2, \dots \dots . . , v_n)$ is a vector in R^n, then the norm of $\mathbf{v}$ (also called the length or the magnitude) is denoted by $||\mathbf{v}||$, and is defined by the formula

$$||\mathbf{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2 \dots \dots . + {v_n}^2}$$

Unit Vectors: A vector of norm 1 is called a unit vector.

If v is any non-zero vector in R^n, then a unit vector in the same direction as v is defined by

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

This process to obtain a unit vector is called normalizing v.

The Standard Unit Vectors: When a rectangular coordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinate axes are called the standard unit vectors.

In R^2, these vectors are denoted by $\mathrm{i} = (1,\ 0)$ & $\mathrm{j} = (0,1)$

and in R^3 by $\mathbf{i} = (1,\ 0,\ 0)$, $\mathbf{j} = (0,\ 1,\ 0)$, & $\mathbf{k} = (0,\ 0,\ 1)$

DISTANCE IN $\boldsymbol{R^n}$: The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in 2-space is

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

And the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space is

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

COMPONENT FORM OF DOT PRODUCT: If $u = (u_1, u_2, \dots \dots \dots, u_n)$ and $v = (v_1, v_2, \dots \dots \dots, v_n)$ are vectors in R^n, then the Dot Product (also called Euclidean Inner Product) of u & v is defined as

$$u.v = u_1v_1 + u_2v_2 + \cdots \dots \dots + u_nv_n$$

The norm (or length) of a vector u in R^n is defined as

$$\|u\| = \sqrt{u.u}$$

ORTHOGONAL VECTORS: Two non-zero vectors u & v in R^n are called Orthogonal or perpendicular (i.e., $\theta = \frac{\pi}{2}$) if $u.v = 0$.

A non-empty set of vectors in R^n is called an Orthogonal Set if all pairs of distinct vectors in the set are Orthogonal.

CROSS- PRODUCT OF VECTORS: If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are vectors in R^3, then the Cross- Product (also called Vector Product) of u & v is defined as

$$u \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\imath}(u_2v_3 - u_3v_2) - \hat{\jmath}(u_1v_3 - u_3v_1) + \hat{k}(u_1v_2 - u_2v_1)$$

NOTE: The vector $u \times v$ is Orthogonal to the vectors u & v both.

CHAPTER-4

LINEAR INDEPENDENCE: Let $S = \{v_1, v_2, \dots \dots .., v_r\}$ is a non-empty set of vectors in a vector space V. If the equation $k_1 v_1 + k_2 v_2 + \cdots \dots \dots + k_r v_r = 0$ has only trivial solution (i.e., $k_1 = 0, k_2 = 0, \dots \dots \dots + k_r = 0$), then S is said to be Linearly Independent.

If there are solutions in addition to trivial solution, then S is said to be Linearly Dependent.

NOTE: If $|A| \neq 0$, then the vectors are Linearly Independent,
and if $|A| = 0$, then the vectors are Linearly Dependent.

BASIS FOR A VECTOR SPACE: If V is any vector space and $S = \{v_1, v_2, \dots \dots .., v_r\}$ is a finite set of vectors in V, then S is called a Basis for V if the following two conditions hold:

a. The set S is Linearly Independent. (i.e., $v_1, v_2, \dots \dots .., v_r$ are Linearly Independent.)
b. The set S spans V. (i.e., every vector in V can be expressed as a linear combination of $v_1, v_2, \dots \dots .., v_r$)

THEOREM: All Bases for a finite-dimensional vector space have the same no. of vectors.

DIMENSION OF VECTOR SPACE: The Dimension of a finite-dimensional vector space V is defined to be the no. of vectors in a basis for V and is denoted by $\dim(V)$. In addition the zero vector space is defined to have dimension zero.

NOTE: $\dim(R^n) = n$, the standard basis has n vectors.
$\dim(P_n) = n + 1$, the standard basis has $n + 1$ vectors.
$\dim(M_{mn}) = mn$, the standard basis has mn vectors.

DIMENSION OF ROW SPACE: The dimension of row space is the no. of basis vectors for the row space of matrix A, thus the dimension of row space is the no. of non-zero rows in Echelon form of A.

THEOREM: The row space and column space of a matrix A have same dimension.

RANK AND NULLITY OF A MATRIX: The common dimension of the row space and column space of a matrix A is called the Rank of A and is denoted by $rank(A)$; the dimension of the null space of A is called the Nullity of A and is denoted by $nullity(A)$.

NOTE (1) The rank of a matrix A can be interpreted as the no. of leading 1's in any row echelon form of the matrix A.
NOTE (2) If A is any $m \times n$ matrix, then $rank(A) \leq \min(m, n)$.

DIMENSION THEOREM FOR MATRICES:
If A is a matrix with n columns, then $rank(A) + nullity(A) = n$.

THEOREM: If A is an $m \times n$ matrix, then
(i) $rank(A) =$ the no. of leading variables in the general solution of $AX = 0$.
(ii) $nullity(A) =$ the no. of parameters in the general solution of $AX = 0$.

THEOREM: If A is any matrix, then $rank(A) = rank(A^T)$.